The Golden Digits International Contest

9th Edition, February 2025

Problem 1. Find all positive integers n such that

$$1^n + 2^n + \cdots + n^n$$

is a prime number.

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Solution: The answer is $n = 2$ only.

Lemma: For k not divisible by $p-1$, the sum $1^k + 2^k + \ldots (p-1)^k$ is divisible by p.

Proof: Let g be a primitive root mod p. Rewriting the numbers in the sum as $g, g^2, \ldots, g^{p-1}$, we need to show that $g^k + g^{2k} + \ldots + g^{(p-1)k}$ is divisible by p. However, this is just a geometric sum and is equal to $g^k \frac{g^{(p-1)k}-1}{g^k-1}$, which is clearly $0 \pmod{p}$ for k not divisible by $p-1$.

Call the number in the problem S_n. Consider a prime p dividing n and split the n powers in $\frac{n}{p}$ numbers of the form $(kp+1)^n + (kp+2)^n + \cdots + ((k+1)p)^n$ for k from 0 to $\frac{n}{p} - 1$. It is clear that all of these numbers are congruent mod p. Thus, if $p^2|n$, as S_n is the sum of $\frac{n}{p}$ congruent numbers, it is divisible by p, thus for S_n to be prime, n must be square-free.

Also, clearly if $1^n + 2^n + \ldots + (p-1)^n$ is divisible by p, S_n is also divisible by p, thus, by the lemma, if p divides n, $p-1$ also does. We will show that all numbers of this form are 1, 2, $2 \cdot 3$, $2 \cdot 3 \cdot 7$ and $2 \cdot 3 \cdot 7 \cdot 43$.

Consider the smallest prime p_1 dividing n, as $p_1 - 1$ also divides n and is smaller than p_1, it is 1, so $p_1 = 2$. Now, in the same way, considering p_2 to be the second smallest prime, $p_2 - 1$ divides 2, so $p_2 = 3$, and similarly $p_3 - 1$ divides 6 so $p_3 = 7$ and $p_4 - 1$ divides 42 so $p_4 = 43$. Now, considering p_5, we know that $p_5 - 1$ divides $42 \cdot 43$. We will show that there is no such prime.

Clearly, if it existed $p_5 - 1$ would have to be divisible by 43, as else we would have had another option for p_4, also $2|p_5 - 1$, which implies $86|p_5 - 1$. As $7 \equiv 1 \pmod{3}$ and $3|86+1 \Rightarrow 3|p_5 - 1$, which implies $7|p_5 - 1$, which finally impies that $42 \cdot 43 + 1 = 13 \cdot 139$ is prime, which is not true.

Thus, we are left to check these values of n. For n equal to 6 and $42 \cdot 43$, S_n is divisible by 13. The former is true because $2S_6 \equiv 1^6 + \ldots + 12^6 \equiv 0 \pmod{1}3$ by the lemma, and the latter is also true by the lemma, as $12 \nmid 42 \cdot 43$ and $13 \mid 42 \cdot 43 + 1$. We can easily see that S_{42} is divisible by 5, so we are only left with $n = 2$.

Remark: We can do the same trick for any primes dividing $n+1$ or $2n+1$ and get the same result.

Problem 2. Let $\triangle ABC$ be an acute triangle. Let I be its incenter, $D = AI \cap (ABC)$ and E, $F \in (BIC)$ such that A, E, F are collinear. Let $E_b \in AB$ such that $EE_b = E_bB$. E_c is defined similarly. Let $K \in E_bE_c$ such that $DK \parallel EF$. Prove that $AK \cap BC \cap DF \neq \emptyset$.

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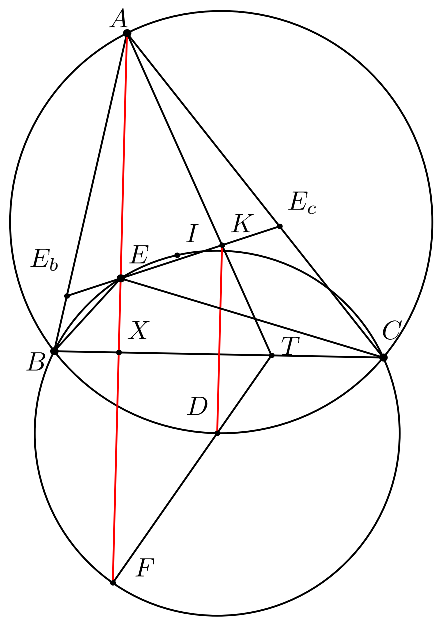

Solution: Let $\alpha_E = \angle CBE$, $\beta_E = \angle ECB$. Similarly define α_F, β_F.

Lemma: E_b, E_c, E are collinear.

Proof: $\angle E_bEB = \angle EBE_b = \angle CBA - \angle CBE$ and similarly $\angle CEE_c = \angle ACB - \angle ECB$ so

$$\angle E_bEB + \angle BEC + \angle CEE_c = \angle BEC + 180 - \angle BAC - 180 + \angle BEC = 2\angle BEC + \angle BAC$$

But $2\angle BEC = 2\angle BIC = 180 - \angle BAC$ so $\angle E_bEB + \angle BEC + \angle CEE_c = 180$ so the lemma is proven.

Ratio lemma: Let ABC be a triangle and D a point on BC. Then

$$\frac{BD}{DC} = \frac{AB \sin BAD}{AC \sin CAD}$$

Proof: By law of sines in $\triangle ABD$ and $\triangle ACD$ we have $\frac{BD}{\sin BAD} = \frac{AB}{\sin ADB}$ and $\frac{CD}{\sin CAD} = \frac{AC}{\sin ADC}$. Dividing these equalities yields the desired result.

We have that $\angle E_cDK = 90 - \angle(DK, CE) = 90 - \angle FEC = 90 - \alpha_F$ and $\angle KDE_b = 90 - \beta_F$ so by ratio lemma in $\triangle D_EcE_b$:

$$\frac{E_cK}{KE_b} = \frac{DE_c \cos \alpha_F}{DE_b \cos \beta_F}$$

Also by ratio lemma in $\triangle AE_bE_c$ we get that

$$\frac{\sin CAK}{\sin BAK} = \frac{KE_c}{E_bK}\frac{AE_b}{AE_c}$$

Let $T = AK \cap BC$. By ratio lemma in $\triangle ABC$ we get:

$$\frac{CT}{TB} = \frac{AC}{AB}\frac{\sin CAK}{\sin BAK}$$

$$\Rightarrow \frac{CT}{BT} = \frac{AC}{AB} \cdot \frac{DE_c \cos \alpha_F}{DE_b \cos \beta_F} \cdot \frac{AE_b}{AE_c}$$

We can easily get that $\angle E_bDB = \beta_E$ so $\angle ADE_b = C - \beta_E$. So by sine law in $\triangle AE_bD$ we get:

$$\frac{AE_b}{E_bD} = \frac{\sin(C - \beta_E)}{\sin \frac{A}{2}}$$

Using the similar relation for E_c we get that:

$$\frac{CT}{BT} = \frac{AC}{AB} \cdot \frac{\sin(C - \beta_E)}{\sin(B - \alpha_E)} \cdot \frac{\cos \alpha_F}{\cos \beta_F}$$

Let $X = AE \cap BC$. Then by ratio lemma in $\triangle ABX$ and $\triangle ACX$ we get that:

$$\frac{AE}{EX} = \frac{AB \sin(B - \alpha_E)}{BX \sin \alpha_E} = \frac{AC \sin(C - \beta_E)}{CX \sin \beta_E}$$

So

$$\frac{CT}{BT} = \frac{CX}{BX} \cdot \frac{\sin \beta_E}{\sin \alpha_E} \cdot \frac{\cos \alpha_F}{\cos \beta_F}$$

Now let $T' = FD \cap BC$. We want to show that $\frac{BT}{CT} = \frac{BT'}{CT'}$.

We have that $\angle CFD = 90 - \alpha_F$ and $\angle DFB = 90 - \beta_F$ so

$$\frac{CT'}{T'B} = \frac{FC \cos \alpha_F}{FB \cos \beta_F}$$

But by ratio lemma in $\triangle BFC$ we have that

$$\frac{CX}{BX} = \frac{FC \sin \alpha_E}{FB \sin \beta_E}$$

But now the equality $\frac{BT}{CT} = \frac{BT'}{CT'}$ is obvious so the problem ends.

Problem 3. A and B play a game on an $n \times n$ board. On each turn, A places a rook on an empty square of the board, and B moves it to a neighboring square (two squares are neighbors if they share a common edge). If all neighboring squares are occupied, the rook remains in place. A wins when there are n rooks on the board that do not attack each other. Determine for which constants $c \in \mathbb{R}$ there exists $c_0 \in \mathbb{R}$ such that A can win in at most $cn + c_0$ moves for all $n \in \mathbb{N}$.

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Solution: Answer: all $c \geq \frac{3}{2}$.

We first show $c = \frac{3}{2}$ is enough for A to win.

A puts a rook 4 times in each $(4k+2, 4k+2)$ square. Then A puts a rook twice in each $(4k+3, 4k+3)$ square.

This way we create a rook-set for the square $n - r_4(n) \times n - r_4(n)$ where $r_4(n)$ is the residue of $n \pmod 4$. Now, in at most 15 moves we can guarantee that B leaves a rook in squares $(n-i, n-i)$ where $i = \overline{1, r_4(n)}$, thus creating a rook set in at most $\frac{3}{2}n + 15$ moves.

We now show that $c < \frac{3}{2}$ does not work.

For the sake of simplicity we first set some terminology. We call a cell occupied if there is a rook on it. We call an odd column cell green and an even column cell white. We also call a column bad if it contains exactly one rook. We say a rook r appeared from a cell c if it was placed by A on c and call a cell's (or rook's) 1, 2, 3, and 4 neighbour the north, east, south and west neighbour respectively. We say a rook (or cell) appeared from 1, 2, 3, and 4 if it appeared from its 1, 2, 3 or 4 neighbour respectively.

We follow the following set of rules:

a) If we can move the rook on a green cell, we do so

b) If A plays on a white cell (x,y) and we are forced to move on a white cell then:

If $(x-1, y+2)$ is occupied then we move the rook upwards. If it isn't, but $(x+1, y+2)$ is occupied then we move it downwards. If not, but $(x-1, y-2)$ is occupied then we move it upwards. If not, but $(x+1, y-2)$ is occupied then we move it downwards. If even that is not the case, we move it to a row that is either 0 or 1 $\pmod 4$.

c) If A plays on a green cell (x, y) and we are forced to move on a white cell then: If at least on of $(x-2, y-1)$, and $(x+2, y-1)$ is occupied then we move it to the left. If not, we move it to the right.

Call R the rook-set formed at the end.

We show that we can assign each white cell $\in R$ (that is not on the last or first 3 rows) an occupied cell that is not in R. We do so inductively, starting from the last even column.

If we have another occupied cell on the same column as w we assign that cell to w. If not, we show that we can assign a cell on an adjacent green column that shares exactly one corner with w. We distinguish 5 cases (in all of them we will suppose for the sake of contradiction that our hypothesis is false):

Case 1: w appeared from w.

w cannot appear from w because that would imply that the cell directly above is occupied.

Case 2: w appeared from 2.

We must have rooks on both B and D. Suppose B is not in R. Then we assign B to w.

If B is also assigned to a cell w', w' must be on the column $y+2$. But since on column y we have only one rook, by condition c cell C is occupied. Then B cannot be assigned to w' so this case is over.

We treat similarly the case where $D \notin R$.

Case 3: w appeared from 3.

In this case, both $D = (x+1, y+1)$ and $F = (x+1, y-1)$ are occupied. If $F \notin R$ the we can assign it to w. So we are left to treat the case $F \in R$.

$D \notin R$ so it must be assigned to $w' = (x+2, y+2)$. Then column $y+2$ is bad. Rule b then implies that w appeared before w'.

w' cannot appear from 4 by condition c. If w' appeared from 1 or 2 then we can assign $(x+1, y+3)$ to it. So w' appeared from 3.

Let $U = (x+3, y+1)$ and $V = (x+3, y+3)$. If $U \notin R$ then we can assign it to w' and hence we can assign D to w. So $U \in R$. Then $V \notin R$. If we cannot assign it to w' then column $y+4$ is bad. Let w" be the cell that has V assigned.

By condition b w' appeared before w". Analyzing the cases as above, we get that w" must appear from 3. But then condition b gives that both $x+2$ and $x+4 \in \{0, 1\} \pmod 4$, absurd.

So this case is proven

Case 4: w appeared from 4.

Both $(x-1, y-1)$ and $(x+1, y-1)$ are occupied, are not assigned to any rook and cannot be simultaneously in R, so we can assign one to w.

Case 5: w appeared from 1.

Analogous to case 3.

With this our induction is complete. So we have at least $\lfloor \frac{n}{2} \rfloor - 7$ cells $\notin R$ so at least $n + \lfloor \frac{n}{2} \rfloor - 7$ in total $\Rightarrow c \geq \frac{3}{2}$ so our proof is complete.